3. PROPERTIES OF THE ELLIPSOID

3.1 Introduction

As discussed in section 1 for many computations in geometric geodesy we deal with the geometry of an ellipsoid of revolution. This ellipsoid is formed by taking an ellipse and rotating it about its minor axis. Let this ellipse be as shown in Figure 3.1.

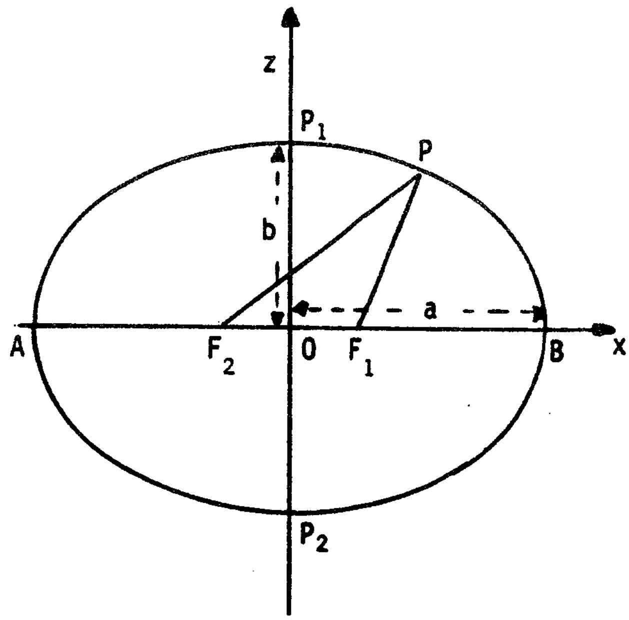

Figure 3.1
The Basic Ellipse

In this figure we have:

F_1, F_2; foci of the ellipse AP_2BP_1;
0 = center of the ellipse;
OA = OB = a = semi-major axis of the ellipse;
OP_1 = OP_2 = b = semi-minor axis of the ellipse;
P_1P_2 is the minor axis of this ellipse while P is an arbitrary point on the ellipse.

From the definition of an ellipse as a locus of a point which moves so that the sum of its distances from two fixed points is a constant we have:

$$F_2P + F_1P = \text{a constant} \tag{3.1}$$

If we let P go to B, and then to A, we can find that:

$$F_2P + F_1P = 2a \tag{3.2}$$

If we now let P go to P_1, and note that $F_2P_1 = F_1P_1$ we must have from equation (3.2) that $F_2P_1 = F_1P_1 = a$, the semi-major axis. This information is shown in the following figure:

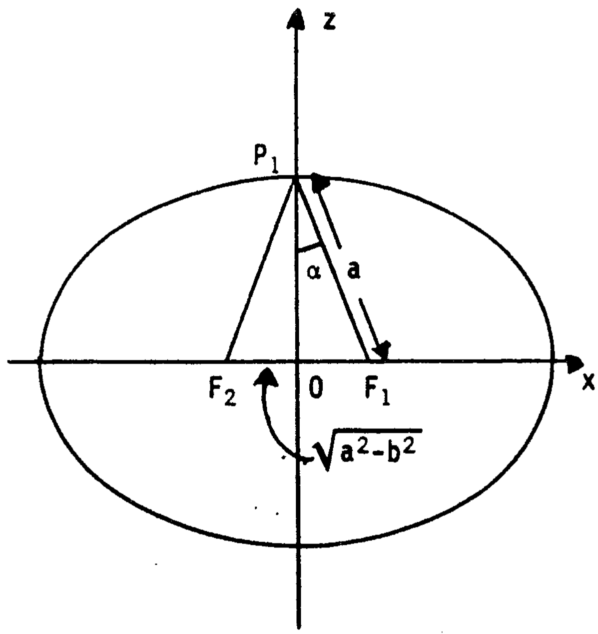

Figure 3.2
Notation for the Ellipse

We are now in a position to define some of the fundamental parameters of this ellipse. We have the following:

1) the polar flattening, f:

$$f \equiv \frac{a - b}{a} \tag{3.3}$$

2) the first eccentricity, e:

$$e \equiv \frac{OF_1}{a} = \frac{\sqrt{a^2 - b^2}}{a}\ ; \quad e^2 = \frac{a^2 - b^2}{a^2} \tag{3.4}$$

3) the second eccentricity, e';

$$e' \equiv \frac{OF_1}{b} = \frac{\sqrt{a^2 - b^2}}{b}\ ; \quad e'^2 = \frac{a^2 - b^2}{b^2} \tag{3.5}$$

4) the angular eccentricity, α (see Figure 3.2); α is the angle at P_1 between the minor axis and a line drawn from P_1 to either F_2 or F_1. We have:

$$\cos\alpha = \frac{b}{a} = 1 - f \tag{3.6}$$

$$\sin\alpha = \frac{OF_1}{a} = e \tag{3.7}$$

$$\tan\alpha = \frac{OF_1}{b} = e' \tag{3.8}$$

5) the linear eccentricity, E;

$$E = ae \tag{3.9}$$

Two other quantities often used are:

$$m \equiv \frac{a^2 - b^2}{a^2 + b^2} \tag{3.10}$$

$$n \equiv \frac{a - b}{a + b} \tag{3.11}$$

In some books the quantity m is designated as e''^2

The basic parameters, a, b, f, e, e', α, m, n are interrelated through equations that can be fairly readily derived. For example, consider the relationship between f and e'. From (3.4) we have:

$$e^2 = 1 - \frac{b^2}{a^2} \tag{3.12}$$

From (3.3):

$$\frac{b}{a} = 1 - f \tag{3.13}$$

which is substituted into (3.12) to find:

$$e^2 = 2f - f^2 \tag{3.14}$$

Other relationships of interest are as follows (Gan'shin, 1967):

$$e^2 = \frac{e'^2}{1+e'^2} = \frac{4n}{(1+n)^2} = \frac{2m}{1+m} \tag{3.15}$$

$$e'^2 = \frac{e^2}{1-e^2} \tag{3.16}$$

$$(1-e^2)(1+e'^2) = 1 \tag{3.17}$$

$$\frac{b}{a} = (1-f) = \sqrt{1-e^2} = \frac{e}{e'} = \frac{1}{\sqrt{1+e'^2}} = \frac{1-n}{1+n} = \sqrt{\frac{1-m}{1+m}} \tag{3.18}$$

$$n = \frac{f}{2-f} = \frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}} \tag{3.19}$$

$$m = \frac{2f-f^2}{1+(1-f)^2} = \frac{2n}{1+n^2} \tag{3.20}$$

In addition it is sometimes convenient to have some series expressions relating certain quantities. For example, we have the following (Gan'shin, 1967):

$$n = (1/2)f + (1/4)f^2 + (1/8)f^3 + (1/16)f^4 + (1/32)f^5 +$$

$$n = (1/4)e^2 + (1/8)e^4 + (5/64)e^6 + (7/128)e^8 + (21/512)e^{10} +$$

$$n = (1/2)m + (1/8)m^3 + (1/16)m^5 +$$

$$m = f + (1/2)f^2 - (1/4)f^4 - (1/4)f^5 +$$

$$m = (1/2)e^2 + (1/4)e^4 + (1/8)e^6 + (1/16)e^8 + (1/32)e^{10} +$$

$$m = 2n - 2n^3 + 2n^5 +$$

$$e'^2 = 2f + 3f^2 + 4f^3 + 5f^4 + 6f^5 +$$

$$e'^2 = 4n + 8n^2 + 12n^3 + 16n^4 + 20n^5 +$$

$$e'^2 = 2m + 2m^2 + 2m^3 + 2m^4 + 2m^5 +$$

The numerical values for these quantities depend on the fundamental definition of a size (a) and shape (usually f) parameter. Many different ellipsoids have been used in the past. Currently the system of constants recommended by the International Association of Geodesy is the Geodetic Reference System 1980 (Moritz, 1980). For this system, quantities of geometric interest are the following:

$$a = 6378137 \text{ m (exact)}$$

$$b = 6356752.3141 \text{ m}$$

$$E = 521854.0097 \text{ m}$$

$$c = 6399593.6259 \text{ m}$$

$$e^2 = 0.00669438002290$$

$$e'^2 = 0.00673949677548$$

$$f = 0.00335281068118$$

$$f^{-1} = 298.257222101$$

$$n = 0.001679220395$$

$$m = 0.003358431319$$

$$Q = 10001965.7293 \text{ m}$$

$$R_1 = 6371008.7714 \text{ m}$$

$$R_2 = 6371007.1810 \text{ m}$$

$$R_3 = 6371000.7900 \text{ m}$$

In the above constants Q is the length of a meridian quadrant, R_1 is the mean radius $(2a+b)/3$, R_2 the radius of a sphere having the same surface area as the ellipsoid, and R_3 is the radius of a sphere having the same volume as the ellipsoid. The derivation of the equations for these quantities will be discussed in later sections.

3.2 Geodetic Coordinates

We first consider a rotational ellipsoid whose center is at O. We define the OZ axis to be the rotational axis of the ellipsoid. The OX axis lies in the equatorial plane and intersects the meridian PEP_1 which is taken as the prime or initial meridian from which longitudes will be measured. The OY axis is in the equatorial plane, perpendicular to the OX axis such that OX, OY, OZ form a right handed coordinate system as seen in Figure 3.3:

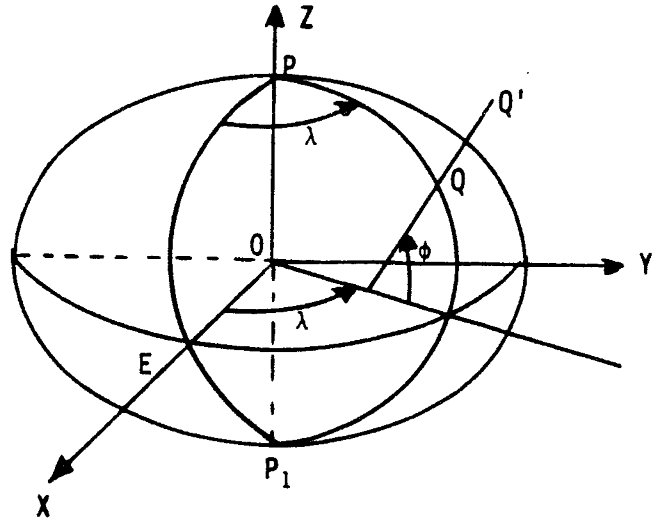

Figure 3.3
Coordinate Systems for the Ellipsoid

An arbitrary point Q or Q' (on or off the surface of the ellipsoid) may then be defined by its X, Y, Z coordinates.

We should note that on a given meridian such as PQP_1 or PEP_1, the longitude is a constant, for any point located on this meridian plane. The geodetic longitude of a point is defined to be the dihedral angle between the planes of the prime meridian (PEP_1) and a meridian (e.g. PQP_1) passing through a given point. Longitudes in this book and for most cases are measured positive eastwards, although there are some cases (e.g. in the United States) where some references consider longitudes measured positive westward.

The geodetic latitude, ϕ, of a point located on the surface of the ellipsoid is defined as the angle between the normal to the ellipsoid at the point and the equatorial plane. For a point located above the surface of the ellipsoid, there are a number of different definitions possible. The simplest one is that it is the angle between the normal to the ellipsoid, passing through this point, and the equatorial plane. This system of coordinates (i.e. ϕ, λ) are called geodetic coordinates. (In some books some references may be found to geographic coordinates which are the same as geodetic coordinates). ϕ and λ form a set of curvilinear coordinates on the surface of the ellipsoid. They allow the description of many properties involved with the surface and curves on the surface.

3.3 The Meridian Ellipse

The meridian ellipse passing through point Q is shown in Figure 3.4 with coordinates axes z and x.

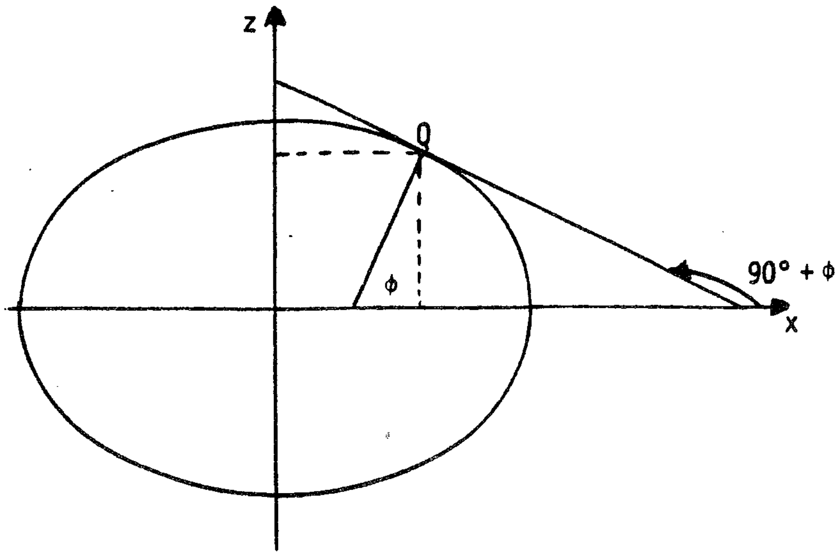

Figure 3.4
The Meridian Ellipse

In addition to geodetic latitude we may also define the reduced latitude β and the geocentric latitude ψ. The reduced latitude (sometimes called the parametric latitude) is the angle at the center of a sphere that is tangent to the ellipsoid along the equator, between the plane of the equator and the radius to the point intersected along the sphere by a straight line perpendicular to the plane of the equator and passing through the point on the ellipsoid whose reduced latitude is being defined. The reduced latitude is shown in Figure 3.5.

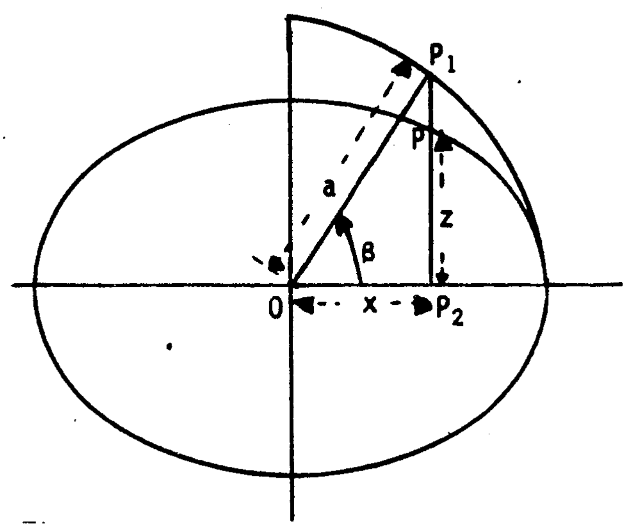

Figure 3.5
The Reduced Latitude

The geocentric latitude is the angle at the center of the ellipse between the plane of the equator and a line to the point whose geocentric latitude is being defined. Note that this definition allows a simple means to define this latitude even though the point may not be located on the surface of the ellipsoid. The geocentric latitude is shown in Figure 3.6.

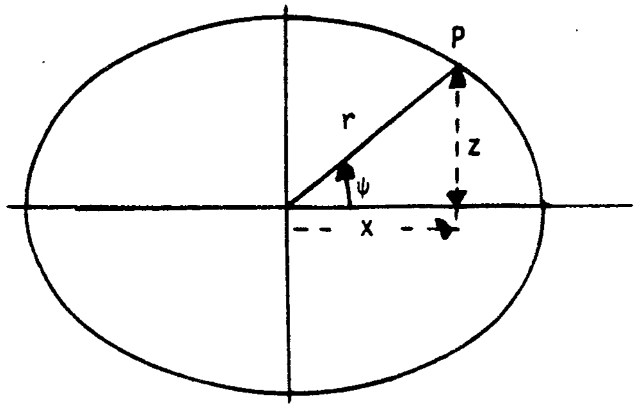

Figure 3.6
The Geocentric Latitude

The z and x coordinates may be computed knowing either ϕ, β, or ψ and the parameters of the ellipsoid. These relationships are useful in deriving expressions that relate the various latitudes.

We first consider the determination of x and z using the reduced latitude β. From Figure 3.5 we write:

$$(OP_2)^2 + (P_2P_1)^2 = a^2 \tag{3.22}$$

The equation of this ellipse may be written:

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{3.23}$$

or with $x = OP_2$ and $z = P_2P$ we have:

$$\frac{(OP_2)^2}{a^2} + \frac{(P_2P)^2}{b^2} = 1 \tag{3.24}$$

Combining (3.22) and (3.24) we have:

$$(OP_2)^2 + (P_2P)^2 \frac{a^2}{b^2} = a^2 = (OP_2)^2 + (P_2P_1)^2 \tag{3.25}$$

Solving for P_2P we find:

$$P_2P = \frac{b}{a} P_2P_1 \tag{3.26}$$

We have from Figure 3.5 that:

$$P_2P_1 = a \sin \beta \tag{3.27}$$

so that the x and z coordinates are:

$$x = OP_2 = a \cos \beta \tag{3.28}$$

$$z = P_2P = b \sin \beta \tag{3.29}$$

To determine x and z using geodetic latitude we note, considering Figure 3.4 that the slope of the tangent line is the tangent of the angle with the positive axis;

$$\frac{dz}{dx} = \tan (90 + \phi) = -\cot \phi = \frac{-\cos \phi}{\sin \phi} \tag{3.30}$$

where $\frac{dz}{dx}$ is the slope of the tangent line. To determine the derivative we rewrite equation (3.23) as follows:

$$b^2x^2 + a^2z^2 = a^2b^2 \tag{3.31}$$

and differentiate to get

$$b^2xdx + a^2zdz = 0 \tag{3.32}$$

or rearranging we have:

$$\frac{dz}{dx} = \frac{-b^2}{a^2} \cdot \frac{x}{z} = \frac{-\cos \phi}{\sin \phi} \tag{3.33}$$

Using equation (3.26) and (3.33) we have:

$$b^2 x \sin\phi = a^2 z \cos\phi \tag{3.34}$$

Squaring both sides we have:

$$b^4 x^2 \sin^2\phi - a^4 z^2 \cos^2\phi = 0 \tag{3.35}$$

We then multiply equation (3.31) by $-b^2\sin^2\phi$, add the result to equation (3.35) and multiply through by -1, and then solve for z to find:

$$z = \frac{b^2 \sin\phi}{(a^2\cos^2\phi + b^2\sin^2\phi)^{\frac{1}{2}}} \tag{3.36}$$

In a similar elimination procedure we find for x:

$$x = \frac{a^2 \cos\phi}{(a^2\cos^2\phi + b^2\sin^2\phi)^{\frac{1}{2}}} \tag{3.37}$$

Using e^2 from equation (3.4) the denominators of equation (3.36) and (3.37) become $a(1-e^2\sin^2\phi)^{\frac{1}{2}}$ so that x and z may be written:

$$x = \frac{a\cos\phi}{(1-e^2\sin^2\phi)^{\frac{1}{2}}} \tag{3.38}$$

$$z = \frac{a(1-e^2)\sin\phi}{(1-e^2\sin^2\phi)^{\frac{1}{2}}} \tag{3.39}$$

At this point it is convenient to introduce and define four new quantities:

$$\begin{aligned} W^2 &\equiv 1 - e^2\sin^2\phi \\ V^2 &\equiv 1 + e'^2\cos^2\phi \\ w^2 &\equiv 1 - e^2\cos^2\beta \\ v^2 &\equiv 1 + e'^2\sin^2\beta \end{aligned} \tag{3.40}$$

Starting from these designations, various other relations may be derived.

$$W^2 = \frac{1}{1+e'^2 \sin^2 \beta}$$

$$V^2 = \frac{1}{1-e^2 \cos^2 \beta} \qquad (3.41)$$

Using W and V in equations (3.38) and (3.39) we can write:

$$x = \frac{a \cos \phi}{W} \qquad (3.42)$$

$$z = \frac{a(1-e^2) \sin \phi}{W} \qquad (3.43)$$

$$x = \frac{c}{V} \cos \phi \qquad (3.44)$$

$$z = \frac{c}{V} \frac{\sin \phi}{(1+e'^2)} \qquad (3.45)$$

where $c = a^2/b$. A geometric interpretation for c will be given later.

A geometric meaning may be attached to W and V by considering elements in Figure 3.7.

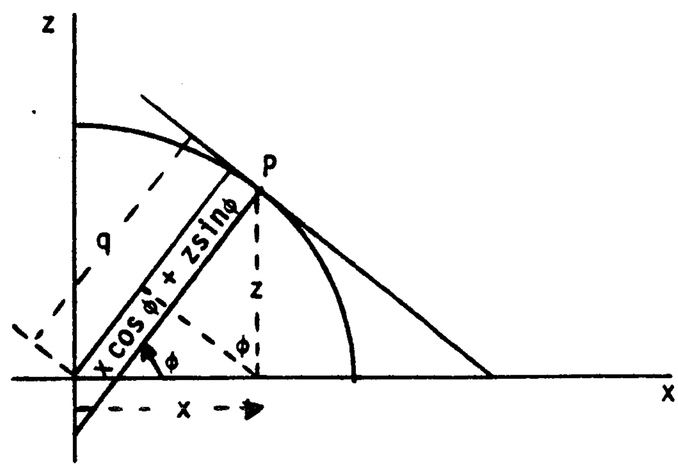

Figure 3.7
A Geometric Interpretation to W and V

In this figure q is a distance measured from the origin to the plane at P (whose geodetic latitude is ϕ) such that the line from the origin is perpendicular to the tangent plane. We have:

$$W^2 = \frac{1}{1+e'^2 \sin^2 \beta}$$

$$V^2 = \frac{1}{1-e^2 \cos^2 \beta} \tag{3.41}$$

Using W and V in equations (3.38) and (3.39) we can write:

$$x = \frac{a \cos \phi}{W} \tag{3.42}$$

$$z = \frac{a(1-e^2) \sin \phi}{W} \tag{3.43}$$

$$x = \frac{c}{V} \cos \phi \tag{3.44}$$

$$z = \frac{c}{V} \frac{\sin \phi}{(1+e'^2)} \tag{3.45}$$

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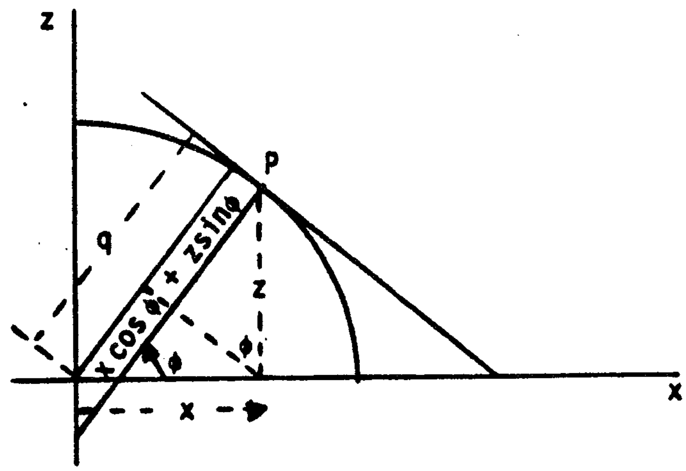

Figure 3.7
A Geometric Interpretation to W and V

In this figure q is a distance measured from the origin to the plane at P (whose geodetic latitude is ϕ) such that the line from the origin is perpendicular to the tangent plane. We have:

$$q = x \cos \phi + z \sin \phi \tag{3.46}$$

Substituting from equations (3.42) and (3.43) we have:

$$q = aW \tag{3.47}$$

From (3.44) and (3.45) we have:

$$q = bV \tag{3.48}$$

We can equate (3.47) and (3.48) to finally write:

$$W = \frac{b}{a} V \tag{3.49}$$

We next turn to the determination of x and z using the geocentric latitude. From Figure 3.6 we write:

$$x = r \cos \psi \tag{3.50}$$

$$z = r \sin \psi \tag{3.51}$$

where r is the geocentric radius.

Clearly we have:

$$r = \sqrt{x^2 + z^2} \tag{3.52}$$

Substituting equation (3.50) and (3.51) into equation (3.23), and solving for r we find:

$$r = \frac{a(1-e^2)^{1/2}}{\sqrt{1-e^2 \cos^2 \psi}} = \frac{b}{\sqrt{1-e^2 \cos^2 \psi}} \tag{3.53}$$

Substituting this value of r back into equations (3.50) and (3.51) we have:

$$x = \frac{a(1-e^2)^{1/2} \cos \psi}{\sqrt{1-e^2 \cos^2 \psi}} \tag{3.54}$$

$$z = \frac{a(1-e^2)^{\frac{1}{2}} \sin \psi}{\sqrt{1-e^2 \cos^2 \psi}} \tag{3.55}$$

We could also obtain expression for the radius vector in terms of geodetic latitude if we substitute equations (3.38) and (3.39) into equation (3.52). We find:

$$r = \frac{a}{W} (1 + e^2 (e^2 - 2) \sin^2 \phi)^{\frac{1}{2}} \tag{3.56}$$

Since the second term on the right hand side of (3.56) is on the order of e^2 it is convenient to obtain a series expression for the radius vector. We first expand the square root term using the binomial series (equation (2.7)) so that:

$$r = \frac{a}{W} (1 + \frac{1}{2} e^2 (e^2 - 2) \sin^2 \phi - \frac{1}{2} e^4 \sin^4 \phi + \ldots) \tag{3.57}$$

Now compute a Maclaurin series expansion (equation (2.4)) for 1/W:

$$\frac{1}{W} = 1 + \frac{e^2}{2} \sin^2 \phi + \frac{3}{8} e^4 \sin^4 \phi + \ldots \tag{3.58}$$

Multiplying (3.57) and (3.58) we find a series expression for r in terms of geodetic latitude:

$$r = a(1 - \frac{e^2}{2} \sin^2 \phi + \frac{e^4}{2} \sin^2 \phi - \frac{5}{8} e^4 \sin^4 \phi + \frac{3}{4} e^6 \sin^4 \phi - \frac{13}{16} e^6 \sin^6 \phi + \ldots) \tag{3.59}$$

The number of terms to retain in such an expression depends on the accuracy desired. Recalling that for the Geodetic Reference System 1980, $a = 6378137$ m, $e^2 = 0.00669\ldots$ the last two terms in equation (3.59) have a maximum value of 0.0008 meters.

3.4 Relationships Between the Various Latitudes

We may use some of the equations previously derived to obtain relationships between the various latitudes described. From Figure 3.6 we write:

$$\tan \psi = \frac{z}{x} \tag{3.60}$$

Substituting for z and x from equations (3.28), (3.29) and (3.38), (3.39) we have:

$$\tan \psi = \frac{b}{a} \tan \beta = (1 - e^2) \tan \phi \tag{3.61}$$

Thus we have:

$$\tan \psi = (1-e^2)^{\frac{1}{2}} \tan \beta = (1 - e^2) \tan \phi \tag{3.62}$$

$$\tan \beta = (1-e^2)^{\frac{1}{2}} \tan \phi \tag{3.63}$$

$$\tan \phi = (1 + e'^2)^{\frac{1}{2}} \tan \beta \tag{3.64}$$

Although these relationships are sufficient to determine one type of latitude given any other, certain procedures are simplified if other relationships are also found. For example, we equate the z coordinate as given in equations (3.29) and (3.43) to obtain:

$$\sin \beta = \frac{(1 - e^2)^{\frac{1}{2}} \sin \phi}{W} = \frac{\sin \phi}{V} \tag{3.65}$$

Equating equations (3.28) and (3.42) dealing with the x coordinate we have

$$\cos \beta = \frac{\cos \phi}{W} \tag{3.66}$$

Other relations of interest include the following:

$$\cos \phi = \frac{\cos \beta}{V} = (1 - e^2)^{\frac{1}{2}} \frac{\cos \beta}{W} \tag{3.67}$$

$$\sin \phi = \frac{\sin \beta}{W} = (1 + e'^2)^{\frac{1}{2}} \frac{\sin \beta}{V} \tag{3.68}$$

Next we turn to the determination of expressions for the determination of the difference between two types of latitude. We first consider closed expressions and then series expressions. We now consider the difference between the geodetic and reduced latitude by writing:

$$\sin (\phi - \beta) = \sin \phi \cos \beta - \cos \phi \sin \beta \tag{3.69}$$

We then substitute values of $\sin \beta$ and $\cos \beta$ from equations (3.65) and (3.66) to obtain after some reductions:

$$\sin(\phi - \beta) = \frac{f \sin 2\phi}{2W} \qquad (3.71)$$

Another closed expression may be written starting from the following identity:

$$\tan(\phi - \beta) = \frac{\tan\phi - \tan\beta}{1+\tan\phi \cdot \tan\beta} \qquad (3.72)$$

Substituting for tan β as a function of tan ϕ we find:

$$\tan(\phi - \beta) = \frac{n \sin 2\phi}{1+n \cos 2\phi} \qquad (3.73)$$

Closed expressions giving a function of $(\phi - \psi)$ as a function of either ϕ or ψ can be derived in closed or series form. To derive a closed expression we write:

$$\tan(\phi - \psi) = \frac{\tan\phi - \tan\psi}{1+\tan\phi \cdot \tan\psi} \qquad (3.74)$$

Substituting (3.61) for tan ψ we can write:

$$\tan(\phi - \psi) = \frac{e^2 \sin 2\phi}{2(1 - e^2 \sin^2\phi)} \qquad (3.75)$$

The derivation of series expressions for the differences of two latitudes can be done using equations (2.11) and (2.12). For example, we may apply this technique to equation (3.63) where $y = \beta$, $p = (1-e^2)^{\frac{1}{2}}$ and $x = \phi$. We find:

$$\phi - \beta = n \sin 2\phi - \frac{n^2}{2} \sin 4\phi + \frac{n^3}{3} \sin 6\phi + \ldots \qquad (3.76)$$

This difference, as a function of β, may be written:

$$\phi - \beta = n \sin 2\beta + \frac{n^2}{2} \sin 4\beta + \frac{n^3}{3} \sin 6\beta + \ldots \qquad (3.77)$$

Using a similar approach the difference between the geodetic and geocentric latitude as a function of ϕ may be written:

$$\phi - \psi = m \sin 2\phi - \frac{m^2}{2} \sin 4\phi + \frac{m^3}{3} \sin 6\phi + \ldots \qquad (3.78)$$

This difference as a function of ψ is:

$$\phi - \psi = m \sin 2\psi + \frac{m^2}{2} \sin 4\psi + \frac{m^3}{3} \sin 6\psi + \ldots \tag{3.79}$$

For the Clarke 1866 ellipsoid (f = 1/294.978698) we have (Adams, 1949):

$$\begin{aligned} \phi - \beta &= 350''.2202 \sin 2\phi - 0''.2973 \sin 4\phi + 0''.0003 \sin 6\phi + \ldots \\ \phi - \psi &= 700''.4385 \sin 2\phi - 1''1893 \sin 4\phi + 0''.0027 \sin 6\phi + \ldots \end{aligned} \tag{3.80}$$

For the ellipsoid of the Geodetic Reference system 1980 we have:

$$\begin{aligned} \phi - \beta &= 346''.3640 \sin 2\phi - 0''.2908 \sin 4\phi + 0''.0003 \sin 6\phi \\ \phi - \psi &= 692''.7262 \sin 2\phi - 1''.1632 \sin 4\phi + 0''.0026 \sin 6\phi \end{aligned} \tag{3.81}$$

We can see that the maximum difference of $\phi - \beta$ is approximately 6' while the maximum difference of $\phi - \psi$ is 12'. This difference occurs close to latitude 45°.

3.5 Radii of Curvature on the Ellipsoid

Consider first a normal to the surface of the ellipsoid at some point. Now take a plane that contains this normal and thus is perpendicular to the tangent plane. This particular plane will cut the surface of the ellipsoid forming a curve which is known as a normal section. The radii of curvature of a normal section will depend on the azimuth of the line. At each point there exist two mutually perpendicular normal sections whose **curvatures** are maximum and minimum. Such normal sections are called principal normal sections.

On the ellipsoid these two normal sections are:

1. the meridional section, a plane passing through the given point and the two poles;

2. the prime vertical section, which is a section through the point and perpendicular to the meridional section at the point.

The radius of curvature in the meridian is designated M and the radius of curvature in the prime vertical direction is designated N.

In order to find the radius of curvature in an arbitrary direction we may utilize Euler's formula:

$$\frac{1}{\rho} = \frac{\cos^2\theta}{\rho_1} + \frac{\sin^2\theta}{\rho_2} \tag{3.82}$$

where ρ is the arbitrary radius of curvature;

θ is the angle measured from the principal section with the largest radius of curvature ρ_1 in a principle normal direction; and

ρ_2 is the radius of curvature in the direction of the other principal normal direction

After examining the N and M values we shall apply equation (3.82) to the ellipsoid case.

3.51 The Radius of Curvature in the Meridian

We first consider the determination of M. We first recall that if we have a plane curve specified as $z = F(x)$, the radius of curvature at a point on the curve is:

$$\rho = \frac{\left[1 + \left(\frac{dz}{dx}\right)^2 \right]^{3/2}}{\frac{d^2z}{dx^2}} \tag{3.83}$$

From equation (3.30) we have:

$$\frac{dz}{dx} = -\cot \phi$$

Then we differentiate this:

$$\frac{d^2z}{dx^2} = \frac{1}{\sin^2\phi}\frac{d\phi}{dx} = \frac{1}{\sin^2\phi}\;\frac{1}{\frac{dx}{d\phi}} \tag{3.84}$$

From equation (3.38) we have:

$$x = \frac{a \cos \phi}{(1-e^2\sin^2\phi)^{1/2}}$$

which is differentiated with respect to ϕ to obtain:

$$\frac{dx}{d\phi} = \frac{-a(1-e^2)\sin \phi}{(1-e^2\sin^2\phi)^{3/2}} \tag{3.85}$$

Using (3.85) in (3.84) we have:

$$\frac{d^2z}{dx^2} = \frac{-(1-e^2\sin^2\phi)^{3/2}}{a \sin^3\phi(1-e^2)} \tag{3.86}$$

Substituting the values of (3.86) and of dz/dx into (3.82) when ρ is now M we find:

$$M = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}} \tag{3.87}$$

where the minus sign has been dropped by convention. Recalling the definitions of W, V, and c, alternate expressions for M are:

$$M = \frac{a(1-e^2)}{W^3} = \frac{c}{V^3} \tag{3.88}$$

We now consider an alternate derivation for M considering Figure 3.8:

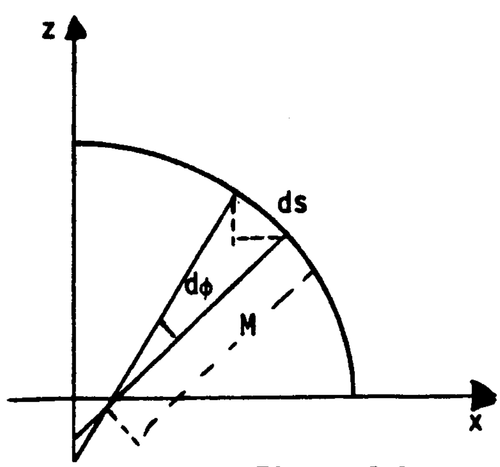

Figure 3.8
A Portion of a Meridian Arc

We have ds a differential distance along a meridian arc; $d\phi$ is the angular separation. Then we regard M as the radius of curvature of the meridian arc so that:

$$ds = Md\phi = \sqrt{dx^2 + dz^2} = dz\sqrt{1 + (\frac{dx}{dz})^2}$$

$$= dz\sqrt{1 + \tan^2\phi} = \frac{dz}{\cos\phi} \qquad (3.89)$$

since:

$$\frac{dz}{dx} = -\cot\phi \quad \text{from equation (3.30)}$$

Then:

$$Md\phi = \frac{dz}{\cos\phi} \quad \text{or}$$

$$M = \frac{1}{\cos\phi}\frac{dz}{d\phi} \qquad (3.90)$$

Using equation (3.39) for z we find:

$$\frac{dz}{d\phi} = \frac{a(1-e^2)\cos\phi}{W^3} \qquad (3.91)$$

which yields from (3.90)

$$M = \frac{a(1-e^2)}{W^3}$$

which is the same as (3.88)

At the equator $\phi = 0$ so that:

$$M_{\phi=0} = a(1-e^2) = a(1-f)^2 \qquad (3.92)$$

At the poles $\phi = \pm 90°$ so that:

$$M_{\phi=90°} = \frac{a(1-e^2)}{(1-e^2)^{3/2}} = \frac{a}{(1-e^2)^{1/2}} = \frac{a}{1-f} = \frac{a^2}{b} = c \qquad (3.93)$$

In this expression, c, as introduced earlier, is seen to be the radius of curvature at the pole.

We could take the ratio:

$$\frac{M_{90}}{M_0} = \frac{a}{1-f} \cdot \frac{1}{a(1-f)^2} = \frac{1}{(1-f)^3}$$

or

$$M_{90} = \frac{M_0}{(1-f)^3} \qquad (3.94)$$

If values of M were tabulated, they could be plotted with respect to an origin at the surface of the reference ellipsoid. The end point of the various M values would fall on a curve as shown in the following diagram.

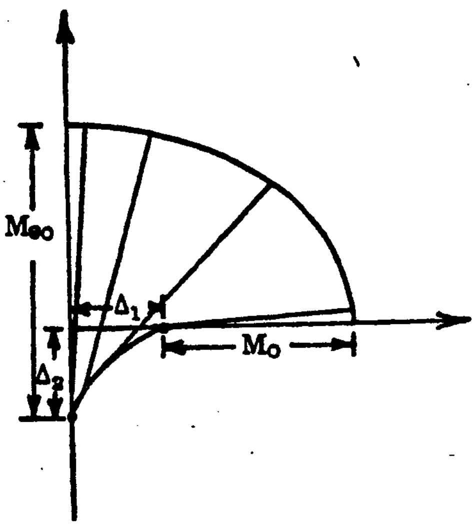

Figure 3.9
Equatorial and Polar Meridian Radii of Curvature

Let us define Δ_1 and Δ_2 as shown in the diagram:

Then:

$$\Delta_1 = a - a(1-f)^2 = a(2f-f^2) = ae^2$$

$$\Delta_1 = ae^2 \tag{3.95}$$

In addition:

$$\Delta_2 = \frac{a}{1-f} - b = \frac{a(2f-f^2)}{(1-f)}$$

$$\Delta_2 = \frac{ae^2}{(1-f)} = \frac{\Delta_1}{(1-f)} \tag{3.96}$$

For the Geodetic Reference System 1980 we have the following values for Δ_1 and Δ_2.

$$\Delta_1 = 42{,}697.67 \text{ m}$$

$$\Delta_2 = 42{,}841.31 \text{ m}$$

3.52 Radius of Curvature in the Prime Vertical

There are several procedures to derive N. One approach is to use the theorem of Meusnier that the radius of the curvature of an inclined section is equal to the curvature radius of a normal section multiplied by the cosine of the angle between these sections. In our case we want to find the radius of curvature of the normal section knowing the radius of curvature of the inclined section. We have:

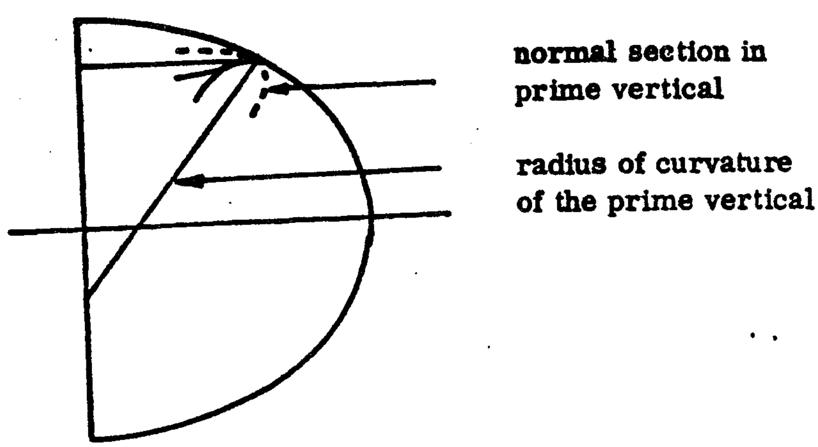

Figure 3.10
Prime Vertical Radius of Curvature

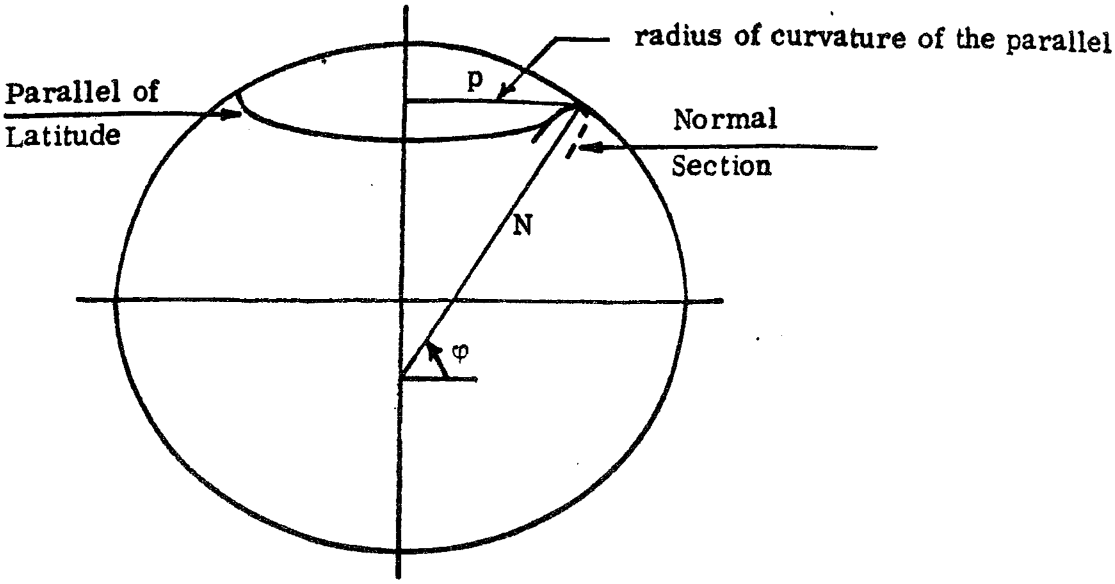

Figure 3.11
Geometry for the Use of Meusnier's Theorem

In the above figure N is the length of the normal line from the surface of the ellipsoid to the intersection of this line with the minor axis.

The radius of curvature of the parallel is p. From the figure:

$$p = N \sin (90 - \phi) = N \cos \phi \tag{3.97}$$

The angle between the prime vertical section and the inclined section is ϕ . Then:

$$p = (\text{prime vertical radius of curvature}) \times \cos \phi \tag{3.98}$$

In equations (3.97) and (3.98) we see that the radius of curvature in the prime vertical direction is N.

An alternate approach is from a geometric argument. To do this we consider the following figure where a prime vertical section has been drawn.

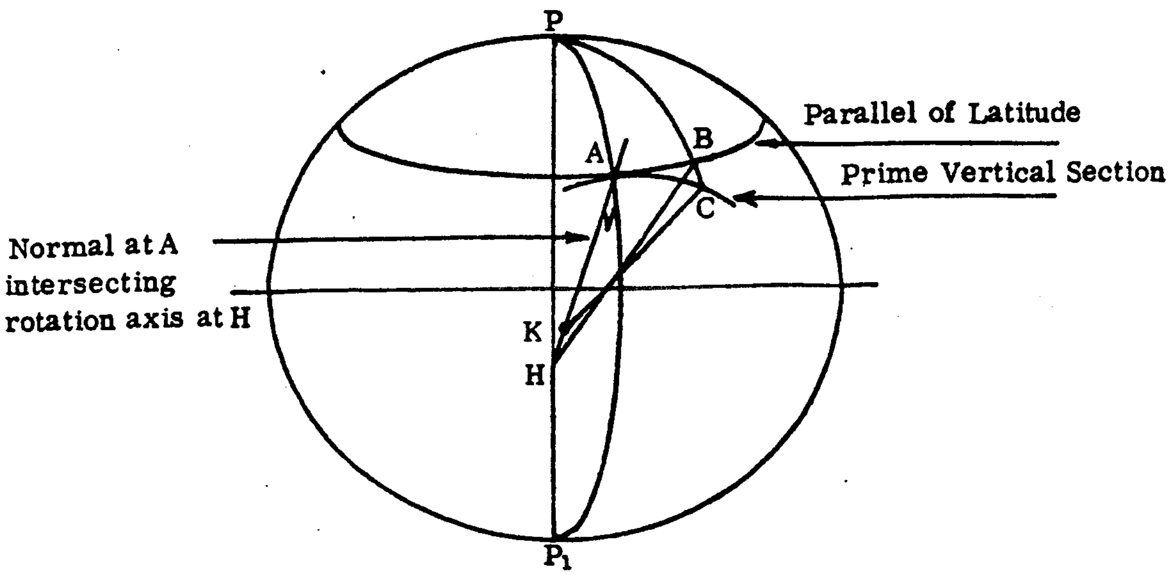

Figure 3.12
Geometric Derivation for N(A)

In this figure, PAP_1 represents the meridian through A. AH is the normal at A, intersecting the rotation axis. B is an arbitrary point on the same parallel as A, while BH is the normal at B intersecting the rotation axis at H. C is a point on the prime vertical section through A and that also lies on the meridian passing through B.

We construct a normal at C that will intersect (at K) the normal from A since AC is a plane curve. We can say that K is the approximate center of curvation of the arc AC. Now let the point B approach point A, so that C will approach A. The intersection of the normals will approach the true center of curvature and CK will approach the true radius of curvature of the arc. Now as C approaches A, C also approaches B so that K will approach H. Thus the radius of curvature of the prime vertical section at A must be the distance from H to A or AH. To compute this radius we consider the meridian ellipse in the following figure.

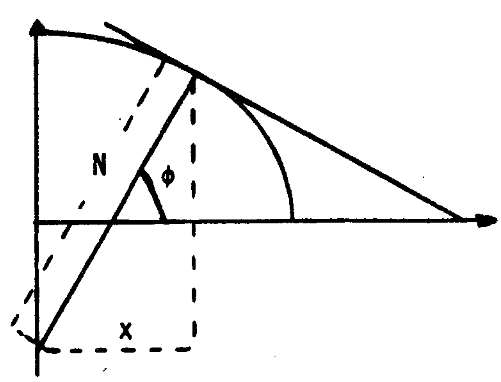

Figure 3.13
Geometric Derivation of N(B)

From the figure we have:

$$x = N \cos \phi$$

Using the expression for x derived previously we can solve for N to find:

$$N = \frac{a}{(1-e^2\sin^2\phi)^{1/2}} = \frac{a}{W} = \frac{c}{V} \tag{3.99}$$

At the equator the prime vertical radius of curvature is:

$$N_{\phi=0} = a \tag{3.100}$$

At the pole:

$$N_{\phi=90°} = \frac{a}{1-f} = \frac{a^2}{b} \tag{3.101}$$

We thus see that M (see 3.92 and 3.39) and N are a minimum for points on the equator. At the pole $M = N = a^2/b = c$ so that both curvatures are the same.

We may find the ratio of N/M by using equations (3.88) and (3.99). We have:

$$\frac{N}{M} = \frac{c}{V} \cdot \frac{V^3}{c} = V^2$$

or:

$$\frac{N}{M} = V^2 = 1 + e'^2\cos^2\phi \tag{3.102}$$

Thus $N \geq M$ where the equality holds at the pole.

3.53 The Radius of Curvature in the Normal Section Azimuth α

Since N is generally greater than M, we associate N with ρ_1 that arose in equation (3.82). If we let α be the azimuth of a line for which we are interested in the curvature, we have $\theta = 90° - \alpha$. If $\rho=R_\alpha$ we then may express equation (3.82) in the following form for the ellipsoid of revolution.

$$\frac{1}{R_\alpha} = \frac{\sin^2\alpha}{N} + \frac{\cos^2\alpha}{M} \qquad (3.103)$$

or:

$$R_\alpha = \frac{MN}{N\cos^2\alpha + M\sin^2\alpha} = \frac{N}{1 + e'^2\cos^2\alpha\cos^2\phi} \qquad (3.104)$$

3.6 Meridian Arc Lengths

We next turn to the computation of lengths of meridian arcs. A differential arc length was written in equation (3.89) as:

$$ds = Md\phi$$

In order to find the length of arc between two points with latitudes ϕ_1 and ϕ_2 we integrate the above equation to write:

$$s = \int_{\phi_1}^{\phi_2} Md\phi = a(1-e^2)\int_{\phi_1}^{\phi_2} \frac{d\phi}{W^3} \qquad (3.105)$$

The integral

$$\int \frac{d\phi}{W^3} = \int (1-e^2\sin^2\phi)^{-3/2}\, d\phi$$

represents an elliptical integral which can not be integrated in elementary functions. Instead the value of $1/W^3$ is expanded in a series and the integration is carried out term by term. First we find the Maclaurin series expansion of $1/W^3$ to be:

$$\frac{1}{W^3} = 1 + \frac{3}{2}e^2\sin^2\phi + \frac{15}{8}e^4\sin^4\phi + \frac{35}{16}e^6\sin^6\phi + \frac{315}{128}e^8\sin^8\phi + \frac{693}{256}e^{10}\sin^{10}\phi --- \qquad (3.106)$$

For ease in integration we replace the powers of $\sin\phi$ by multiple angle equivalents as given in equation (2.18) to find:

$$\frac{1}{W^3} = A - B\cos 2\phi + C\cos 4\phi - D\cos 6\phi + E\cos 8\phi - F\cos 8\phi - F\cos 10\phi + --- \quad (3.107)$$

where the coefficients A, B, etc. have the following meaning:

$$
\begin{aligned}
A &= 1 + \frac{3}{4}e^2 + \frac{45}{64}e^4 + \frac{175}{256}e^6 + \frac{11025}{16384}e^8 + \frac{43659}{65536}e^{10} + \ldots \\
B &= \frac{3}{4}e^2 + \frac{15}{16}e^4 + \frac{525}{512}e^6 + \frac{2205}{2048}e^8 + \frac{72765}{65536}e^{10} + \ldots \\
C &= \frac{15}{64}e^4 + \frac{105}{256}e^6 + \frac{2205}{4096}e^8 + \frac{10395}{16384}e^{10} + \ldots \\
D &= \frac{35}{512}e^5 + \frac{315}{2048}e^8 + \frac{31185}{131072}e^{10} + \ldots \\
E &= \frac{315}{16384}e^8 + \frac{3465}{65536}e^{10} + \ldots \\
F &= \frac{693}{131072}e^{10} + \ldots
\end{aligned}
\quad (3.108)
$$

We can now substitute (3.107) into (3.105) to write:

$$
\begin{aligned}
s &= a(1-e^2) \int_{\phi_1}^{\phi_2} (A - B\cos 2\phi + C\cos 4\phi)\, d\phi + --- \\
&= a(1-e^2) \left[\int_{\phi_1}^{\phi_2} A\, d\phi - B \int_{\phi_1}^{\phi_2} \cos 2\phi\, d\phi + C \int_{\phi_1}^{\phi_2} \cos 4\phi\, d\phi\right] + --- \\
&= a(1-e^2) \left[A\phi \Big|_{\phi_1}^{\phi_2} - \frac{B}{2}\sin 2\phi \Big|_{\phi_1}^{\phi_2} + \frac{C}{4}\sin 4\phi \Big|_{\phi_1}^{\phi_2}\right] + --- \quad (3.109)
\end{aligned}
$$

$$
\begin{aligned}
s = a(1-e^2) \Big[&A(\phi_2 - \phi_1) - \frac{B}{2}(\sin 2\phi_2 - \sin 2\phi_1) + \frac{C}{4}(\sin 4\phi_2 - \sin 4\phi_1) \\
&- \frac{D}{6}(\sin 6\phi_2 - \sin 6\phi_1) + \frac{E}{8}(\sin 8\phi_2 - \sin 8\phi_1) - \frac{F}{10}(\sin 10\phi_2 - \sin 10\phi_1)\Big] + ---
\end{aligned}
\quad (3.110)
$$

This equation may be written in an alternate form by using equation (2.22)

In this case $X = \phi_2$, $Y = \phi_1$, so that:

$$\sin n\phi_2 - \sin n\phi_1 = 2 \cos n \left(\frac{\phi_1+\phi_2}{2}\right) \sin \frac{n}{2} (\phi_2-\phi_1) \tag{3.111}$$

Letting:

$$\phi_m = \frac{\phi_1+\phi_2}{2}$$

and:

$$\Delta\phi = \phi_2 - \phi_1$$

we can write specific values of (3.111) as:

$$\begin{aligned}
\sin2\phi_2 - \sin2\phi_1 &= 2\cos2\phi_m\sin\Delta\phi \\
\sin4\phi_2 - \sin4\phi_1 &= 2\cos4\phi_m\sin2\Delta\phi \\
\sin6\phi_2 - \sin6\phi_1 &= 2\cos6\phi_m\sin3\Delta\phi
\end{aligned} \tag{3.112}$$

and so forth. Equation (3.112) may then be substituted into equation (3.110) to yield:

$$s = a(1-e^2)[A\Delta\phi - B\cos2\phi_m\sin\Delta\phi + \frac{C}{2}\cos4\phi_m\sin2\Delta\phi - \frac{D}{3}\cos6\phi_m\sin3\Delta\phi$$

$$+ \frac{E}{4}\cos8\phi_m\sin4\Delta\phi - \frac{F}{5}\cos10\phi_m\sin5\Delta\phi + ---] \tag{3.113}$$

In order to compute the length of the meridian arc from the equator to an arbitrary latitude ϕ we let ϕ_1 equal zero and ϕ_2 equal ϕ in equation (3.110). We then find (with $s = S_\phi$):

$$S_\phi = a(1-e^2)[A\phi - \frac{B}{2}\sin2\phi + \frac{C}{4}\sin4\phi - \frac{D}{6}\sin6\phi + \frac{E}{8}\sin8\phi - \frac{F}{10}\sin10\phi] + --- \tag{3.114}$$

Helmert (1880) carried out an alternate derivation for the meridian arc length in which the expansion parameter is n instead of e^2. In this case a faster convergence of the series is obtained. We have:

$$S_\phi = \frac{a}{1+n}[a_0\phi - a_2\sin 2\phi + a_4\sin 4\phi - a_6\sin 6\phi + a_8\sin 8\phi] \quad (3.115)$$

where:

$$\begin{aligned} a_0 &= 1 + \frac{n^2}{4} + \frac{n^4}{64} \\ a_2 &= \frac{3}{2}\left(n - \frac{n^3}{8}\right) \\ a_4 &= \frac{15}{16}\left(n^2 - \frac{n^4}{4}\right) \\ a_6 &= \frac{35}{48}n^3 \\ a_8 &= \frac{315}{512}n^4 \end{aligned} \quad (3.116)$$

To achieve an accuracy in S_ϕ of 0.1 mm from the equator to the pole, it is sufficient to set a_8 to zero, and neglect terms of n^4 in the a_i coefficients.

Using either equation (3.114) or equation (3.115) it is a simple matter to find the arc distance from the equator to the pole by letting $\phi = 90°$. From equations (3.114) and (3.115) we have:

$$S_{\phi=90°} = a(1-e^2)A\frac{\pi}{2} = \frac{aa_0}{1+n}\frac{\pi}{2} \quad (3.117)$$

For the Geodetic Reference System 1980 we have the following constants associated with the meridian arc computation:

$$\begin{aligned} A &= 1.00505250181 \\ B &= 0.00506310862 \\ C &= 0.00001062759 \\ D &= 0.00000002082 \\ E &= 0.00000000004 \\ F &= 0.00000000000 \\ a_0 &= 1.00000070495 \\ a_2 &= 0.00251882970 \\ a_4 &= 0.00000264354 \\ a_6 &= 0.00000000345 \\ a_8 &= 0.00000000000 \end{aligned} \quad (3.118)$$

The evaluation of (3.117) gives for the quadrant of the ellipsoid of GRS80: 10,001,965.7293 m.

For some applications it is convenient to modify equations such as (3.113) so that equations valid for shorter length lines may be obtained. We make, in (3.113), the following substitution:

$$\sin\Delta\phi = \Delta\phi - \frac{\Delta\phi^3}{6}$$

$$\sin 2\Delta\phi = 2\Delta\phi$$

Retaining basic terms to $\cos 4\phi_m \sin 2\Delta\phi$ but making approximations consistent with the length of lines that the expressions are to be valid for we find (Zakatov, 1962, p. 27):

$$s = a\Delta\phi[1 - (\frac{1}{4} + \frac{3}{4}\cos 2\phi_m)\, e^2 - (\frac{3}{64} + \frac{3}{16}\cos 2\phi_m - \frac{15}{64}\cos 4\phi_m)\, e^4$$

$$+ \frac{1}{8} e^2 \Delta\phi^2 \cos 2\phi_m] \qquad (3.119)$$

Equation (3.119) is accurate for lines with $\Delta\phi$ =5° (length = 556 km) to .03 m. If $\Delta\phi$ = 10° (length = 1100 km) the error is .07 m.

For even shorter lines, simplified equations may be derived. If we let M_m be the meridian radius of curvature at the mean latitude (i.e. ϕ_m) of the line, it can be shown (Zakatov, p. 27) that:

$$s = M_m \Delta\phi\ [1 + \frac{1}{8} e^2 \Delta\phi^2 \cos 2\phi_m] \qquad (3.120)$$

For $\Delta\phi$ = 5° the error in this equation is 0.03 m. For lines less than 45 km in length, the term in brackets in equation (3.120) may be dropped so that for this shorter distance we have:

$$s = M_m \Delta\phi \qquad (3.121)$$

3.7 Length of a Parallel Arc

We next turn to the computation of the length of arc on the ellipsoid between two points having longitudes λ_2 and λ_1 situated on the same parallel. The distance, L, desired is indicated in Figure 3.14.

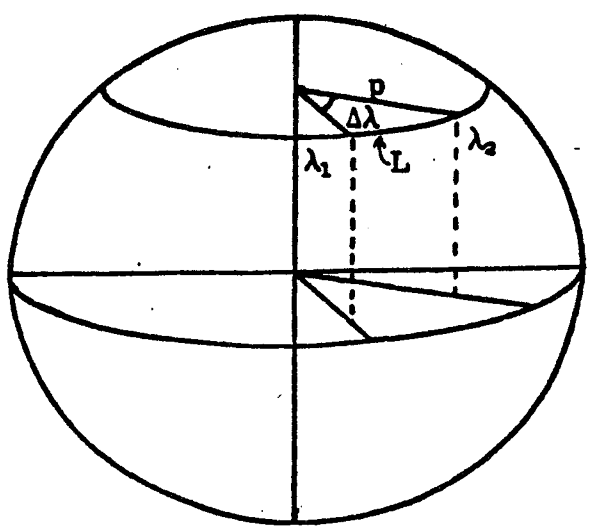

Figure 3.14
Parallel Arc Length

We recall from equation (3.97) that the length of the parallel radius p is $N\cos\phi$. Thus from the figure:

$$L = p\Delta\lambda = N\cos\phi\Delta\lambda \tag{3.122}$$

where $\Delta\lambda$ is in radians.

3.8 Calculation of Areas on the Surface of an Ellipsoid

We consider the area, on the ellipsoid, bounded by given meridians and parallels. To do this we first consider the differential figure shown in Figure 3.15.

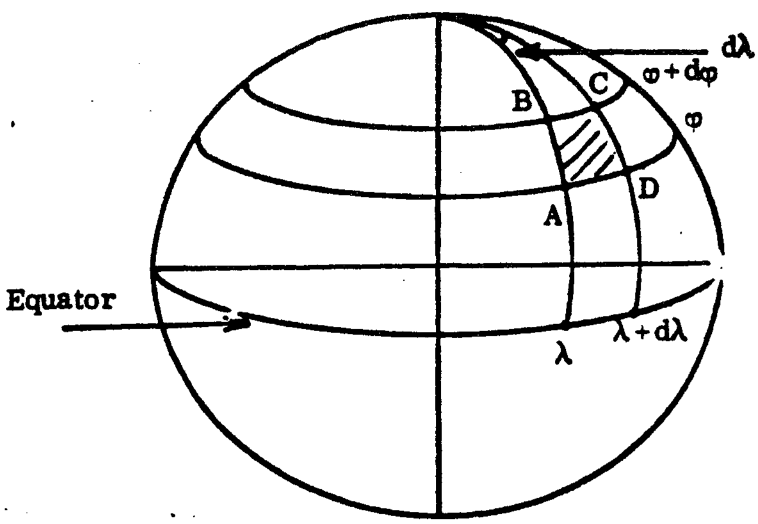

Figure 3.15
Area Element on the Ellipsoid

from the differential figure ABCD we have:

$$AB = CD = Md\phi$$
$$AD = BC = N\cos\phi d\lambda \tag{3.123}$$

Letting the area of the differential figure be dZ we have:

$$dZ = AD \cdot AB = MN\cos\phi d\phi d\lambda \tag{3.124}$$

The area between meridians designated by λ_2 and λ_1, and parallels designated by ϕ_2 and ϕ_1 is:

$$Z = \int dz = \int_{\phi_1}^{\phi_2} \int_{\lambda_1}^{\lambda_2} MN\cos\phi d\phi d\lambda \tag{3.125}$$

Integrating with respect to λ we have:

$$Z = (\lambda_2 - \lambda_1) \int_{\phi_1}^{\phi_2} MN\cos\phi d\phi \tag{3.126}$$

In order to evaluate the integral we substitute for MN to write:

$$\int_{\phi_1}^{\phi_2} MN\cos\phi d\phi = b^2 \int_{\phi_1}^{\phi_2} \frac{\cos\phi}{(1-e^2\sin^2\phi)^2} d\phi \tag{3.127}$$

The integral occuring in (3.127) may be given in closed form as follows (Bagratuni, 1967, p. 59):

$$b^2\int_{\phi_1}^{\phi_2} \frac{\cos\phi \, d\phi}{(1-e^2\sin^2\phi)^2} = \frac{b^2}{2}\left[\frac{\sin\phi}{1-e^2\sin^2\phi} + \frac{1}{2e}\ln\frac{1+e\sin\phi}{1-e\sin\phi}\right]\Bigg|_{\phi_1}^{\phi_2} \tag{3.128}$$

Therefore equation (3.126) may be written:

$$Z = \frac{(\lambda_2-\lambda_1)b^2}{2}\left[\frac{\sin\phi}{1-e^2\sin^2\phi} + \frac{1}{2e}\ln\frac{1+e\sin\phi}{1-e\sin\phi}\right]\Bigg|_{\phi_1}^{\phi_2} \tag{3.129}$$

As a special case of equation (3.129) we compute the area on the ellipsoid from the equator to latitude ϕ, completely around the ellipsoid. Then $(\lambda_2-\lambda_1) = 2\pi$, $\phi_1 = 0$ and $\phi_2 = \phi$ so that equation (3.129) becomes:

$$Z_{0-\phi} = \pi b^2\left[\frac{\sin\phi}{1-e^2\sin^2\phi} + \frac{1}{2e}\ln\frac{1+e\sin\phi}{1-e\sin\phi}\right] \tag{3.130}$$

If we are interested in the area of the half ellipsoid we let $\phi = 90°$ in equation (3.130) to write:

$$Z_{0-90°} = \pi b^2\left[\frac{1}{1-e^2} + \frac{1}{2e}\ln\frac{1+e}{1-e}\right] \tag{3.131}$$

In order to find the area of the whole ellipsoid multiply equation (3.131) by two.

In some cases, it may be more convenient to integrate equation (3.127) using an expansion of the kernel into a series and its subsequent term by term integration. We first write:

$$\frac{\cos\phi}{(1-e^2\sin^2\phi)^2} = \cos\phi + 2e^2\cos\phi\sin^2\phi + 3e^4\cos\phi\sin^4\phi$$

$$+ 4e^6\cos\phi\sin^6\phi + \ldots \tag{3.132}$$

Equation (3.132) may be used in equation (3.127) which is used in (3.126) to find:

$$Z = b^2(\lambda_2-\lambda_1)[\sin\phi + \frac{2}{3}e^2\sin^3\phi + \frac{3}{5}e^4\sin^5\phi + \frac{4}{7}e^6\sin^7\phi + \ldots]\Big|_{\phi_1}^{\phi_2} \tag{3.133}$$

If $(\lambda_2 - \lambda_1) = 2\pi$, and $\phi_1 = 0°$, we find an equation from (3.133) corresponding to (3.130) as:

$$Z_{0-\phi} = 2\pi b^2[\sin\phi + \frac{2}{3}e^2\sin^3\phi + \frac{3}{5}e^4\sin^5\phi + \frac{4}{7}e^6\sin^7\phi + \frac{5}{9}e^8\sin^9\phi + ---] \tag{3.134}$$

The area of the whole ellipsoid, Σ, may be found by letting $\phi = 90°$ in equation (3.134) and doubling the result. We find:

$$\Sigma = 4\pi b^2 \left[1 + \frac{2}{3}e^2 + \frac{3}{5}e^4 + \frac{4}{7}e^6 + \frac{5}{9}e^8 + \frac{6}{11}e^{10} + \ldots\right] \tag{3.135}$$

Equation (3.135) will be useful in a subsequent section

The area of the ellipsoid of GRS80 is 510065621.7 km².

3.9 Radii of Spherical Approximation to the Earth or Mean Radius of the Earth as a Sphere

In some applications it is convenient to let the earth be a sphere rather than an ellipsoid. It is then necessary to find a suitable radius, R, of the sphere to be used. A suitable radius may be defined in several ways that are outlined in the following sections.

3.91 The Gaussian Mean Radius

The Gaussian mean radius is defined to be the integral mean value of R taken over the azimuth varying from 0° to 360°. Designating such a radius as R we have:

$$R = \frac{1}{2\pi}\int_0^{2\pi} R_\alpha d\alpha = \frac{1}{2\pi}\int_0^{2\pi} \frac{MN}{N\cos^2\alpha + M\sin^2\alpha}\, d\alpha \tag{3.136}$$

$$R = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\frac{M}{\cos^2\alpha}}{1+\frac{M}{N}\tan^2\alpha} d\alpha \qquad (3.137)$$

Removing $\sqrt{MN}$, equation (3.137) may be written as:

$$R = \frac{2}{\pi} \sqrt{MN} \int_0^{\frac{\pi}{2}} \frac{\sqrt{\frac{M}{N}} \frac{d\alpha}{\cos^2\alpha}}{1+\left(\sqrt{\frac{M}{N}}\tan\alpha\right)^2} \qquad (3.138)$$

If we let $t = \sqrt{(M/N)}\tan\alpha$, and change the limits, equation (3.138) may be written as:

$$R = \frac{2}{\pi} \sqrt{MN} \int_0^\infty \frac{dt}{1+t^2} \qquad (3.139)$$

which upon integration yields:

$$R = \sqrt{MN} = \frac{a\sqrt{1-e^2}}{1-e^2\sin^2\phi} \qquad (3.140)$$

3.92 Radius of a Sphere Having the Mean of the Three Semi Axes of the Ellipsoid

We let:

$$R_m = \frac{a+a+b}{3} \qquad (3.140)$$

Substituting for b and expanding we have:

$$R_m = a\left[\frac{2}{3} + \frac{\sqrt{1-e^2}}{3}\right] = a\left[\frac{2}{3} + \frac{1}{3}\left(1 - \frac{e^2}{2} + \ldots\right)\right]$$

$$R_m = a\left(1 - \frac{e^2}{6} - \frac{e^4}{24} - \frac{e^6}{48} \ldots\right) \qquad (3.141)$$

3.93 Spherical Radius of Sphere Having the Same Area as the Ellipsoid

To find such a radius we set the area of a sphere equal to the area of the ellipsoid letting R_A be the radius of the sphere. Then:

$$4\pi R_A^2 = \Sigma \qquad (3.142)$$

We find R_A from:

$$R_A = \sqrt{\frac{\Sigma}{4\pi}} \tag{3.143}$$

Using equation (3.135) we find:

$$R_A = a(1 - \frac{e^2}{6} - \frac{17}{360} e^4 - \frac{67}{3024} e^6 + \ldots) \tag{3.144}$$

<u>3.94 Radius of a Sphere having the Same Volume as the Ellipsoid</u>

The volume of a sphere, V_S, is expressed as:

$$V_S = \frac{4}{3} \pi R_V^3 \tag{3.145}$$

where R_V is the radius of the sphere. The volume of an ellipsoid is expressed as:

$$V_E = \frac{4}{3} \pi a^2 b \tag{3.146}$$

Equating equation (3.145) and (3.146) we find:

$$R_v = \sqrt[3]{a^2 b} \tag{3.147}$$

Substituting for b we have:

$$R_v = a(1-e^2)^{1/6} \tag{3.148}$$

Expanding $(1-e^2)^{1/6}$ into a Maclaurin series, equation (3.148) can be expressed as:

$$R_v = a(1 - \frac{e^2}{6} - \frac{5}{72} e^4 - \frac{55}{1296} e^6 \ldots) \tag{3.149}$$

For the parameters of the Geodetic Reference System 1980 we have:

R_m = 6371008.7714 m

R_A = 6371007.1810 m

R_V = 6371000.7900 m

Clearly the distinction between these radii is numerically small. For most applications one might use simply 6371 km. An alternate technique for a spherical radius is to take the Gaussian mean radius at a specified latitude.

3.10 Space Rectangular Coordinates

In discussions connected with Figure 3.3 we defined the X, Y, Z axis. Now we consider the computation of the X, Y, Z coordinates of a point located at a geometric height, h, above the reference ellipsoid. The geometric height is measured along the ellipsoidal normal. To start we consider the meridian ellipse shown in Figure 3.16.

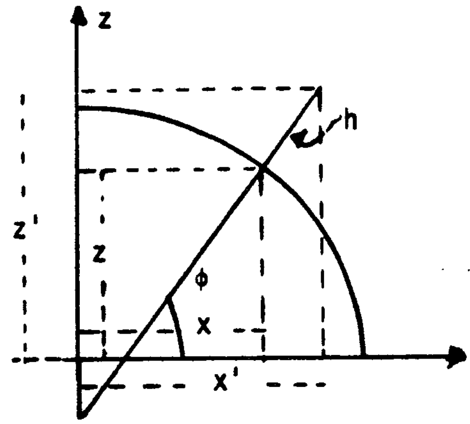

Figure 3.16
The Geometry of a Point Above a Meridian Ellipse

We have:

$$x' = x + h \cos \phi$$
$$z' = z + h \sin \phi \qquad (3.150)$$

where x and z are given by equations (3.42) and (3.44).

The space rectangular coordinates, as can be seen from Figure 3.16 can be related to x' and z' as follows:

$$X = x' \cos \lambda$$
$$Y = x' \sin \lambda \qquad (3.151)$$
$$Z = z'$$

Using equations (3.42) and (3.43) and the expression for x' and z', we have:

$$
\begin{aligned}
X &= (N + h) \cos\phi\cos\lambda \\
Y &= (N + h) \cos\phi\sin\lambda \\
Z &= (N(1 - e^2) + h)\sin\phi
\end{aligned}
\tag{3.152}
$$

where $N = a/W$. A problem to be discussed later will be the computation of ϕ , λ , and h given the space rectangular coordinates X, Y, Z.

3.11 An Alternate Form for the Equation of the Ellipsoid

We have previously written the equation of an ellipse (see equation 3.23) in the form:

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$$

where x is the coordinate measured parallel to the semi-major axis and z is measured parallel to the semi-minor axis. The equation of the ellipsoid can be written in a similar fashion as:

$$\frac{X^2}{a^2} + \frac{Y^2}{a^2} + \frac{Z^2}{b^2} = 1 \tag{3.153}$$

where X, Y, Z are the space rectangular coordinates for the points on the ellipsoid.

An alternate form to (3.153) has been described by Tobey (1928). We first define the axes x', y', and z' at a point P on the surface of the ellipsoid. x' is tangent to the ellipsoid towards the pole, y' is tangent to the ellipsoid in an easterly direction and z' is normal to the ellipsoid, positive towards the center. This system is shown in Figure 3.17.

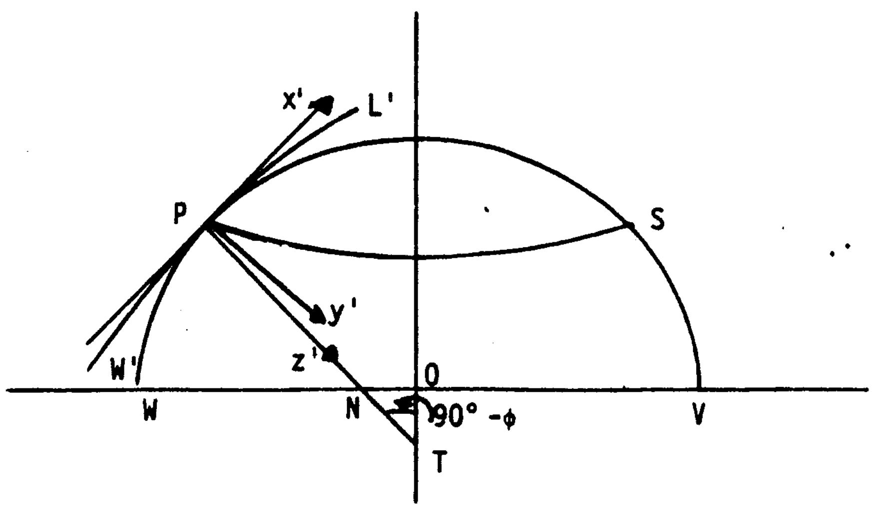

Figure 3.17
A Local Coordinate System on the Ellipsoid

Using the notation of Tobey we indicate the meridian section of the ellipse as WPLSV. The normal from P to the minor axis is the prime vertical radius of curvature, N, and ϕ is the geodetic latitude of point P.

Define a sphere of radius N that has its center at T and is thus tangent to the ellipsoid at P and to all the points on the parallel PS. The equation of this circle in the meridian plane is:

$$x'^2 + z'^2 - 2Nz' = 0 \qquad (3.154)$$

where the origin is at P. The corresponding equation for the tangent sphere would be:

$$x'^2 + y'^2 + z'^2 - 2Nz' = 0 \qquad (3.155)$$

The meridian ellipse is the curve which is tangent at P where the line $x'\cos\phi - z'\sin\phi = 0$ cuts the circle $x'^2 + z'^2 - 2Nz' = 0$. Therefore the equation of the meridian ellipse in this local coordinate system takes the form:

$$x'^2 + z'^2 - 2Nz' + \delta(x'\cos\phi - z'\sin\phi)^2 = 0 \qquad (3.156)$$

An ellipsoid equation must reduce to (3.156) when $y' = 0$. Therefore the general equation for an ellipsoid could be written as:

$$x'^2 + z'^2 - 2Nz' + \delta(x'\cos\phi - z'\sin\phi)^2 + f(y') = 0 \qquad (3.157)$$

Letting $\delta = 0$ and comparing (3.157) with (3.155) we have $f(y') = y'^2$ so that the equation of the ellipsoid will be:

$$x'^2 + y'^2 + z'^2 - 2Nz' + \delta(x'\cos\phi - z'\sin\phi)^2 = 0 \qquad (3.158)$$

Tobey (ibid. Proposition I) shows that $\delta = e'^2$ which was defined earlier. Equation (3.158) is viewed as an alternate form to (3.153) for the equation of the rotational ellipsoid.